Weighted Zygmund estimates for mixed fractional integration

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Abstract: *In the presented work for operators the mixed fractional integration character of improvement of smoothness in comparison with smoothness of density* $\varphi(x, y)$ *with weight* $\rho(x, y)$ *in case of its any continuity modulus is found out* $\omega(\rho\varphi; x, y)$. *Zygmund type estimates are received. We consider operators of mixed fractional integration in weighted generalized Hölder spaces of a function of two variables defined by a mixed modulus of continuity.*

Keywords: *function of two variables, fractional integral, the mixed fractional integral, the mixed continuity modulus, weighted function, Zygmund type estimates.*

1. Introduction

An important stage in the study of fractional integro-differentiation of functions from generalized Hölder spaces (see [1] - [4], [7]) is obtaining estimates of Zygmund type; Estimate of the modulus of continuity of a fractional integral (fractional derivative) in terms of the modulus of continuity of the original function.

The main thrust of the work is to obtain an estimate of the Zygmund type that majorizes the mixed modulus $\omega(\rho I^{\alpha,\beta}_{a+,c+}\varphi; h,\eta)$ of continuity of a mixed fractional integral with the weight of integral constructions from the mixed modulus of continuity $\omega(\rho\varphi; h,\eta)$ of its density $\varphi(x, y)$ with weight $\rho(x, y)$. These Zygmund-type estimates and action theorems directly affect the character of the improvement of the modulus of continuity by a mixed fractional integration $I^{\alpha,\beta}_{a+,c+}\varphi$ of order (α, β)

$$\left(I^{\alpha,\beta}_{0+,0+}\varphi\right)(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y \frac{\varphi(t,\tau)\,dt\,d\tau}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}, \qquad (1.1)$$

where $x, y > 0$, $\alpha, \beta \in (0, 1)$ have not been studied.

It should be emphasized that the presence of weight significantly affects the nature of the Zygmund type evaluation. This was known in the case of Zygmund type estimates for fractional integrals of functions of one variable.

This paper is devoted to the study of certain properties of the mixed fractional integral (1.1) in weighed generalized Hölder spaces of a function of two variables defined by a mixed modulus of continuity.

We consider the operator (1.1) in $Q = \{(x, y):\ 0 < x < b,\ 0 < y < d\}$.

2. Preliminary information and notations

For a continuous function $\varphi(x, y)$ on $\mathbf{R}^2$ we introduce the notation

$$\left(\overset{1,0}{\Delta}_h \varphi\right)(x, y) = \varphi(x+h, y) - \varphi(x, y);$$

$$\left(\overset{0,1}{\Delta}_\eta \varphi\right)(x, y) = \varphi(x, y+\eta) - \varphi(x, y);$$

$$\left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x, y) = \varphi(x+h, y+\eta) - \varphi(x+h, y) - \varphi(x, y+\eta) + \varphi(x, y),$$

so that

$$\varphi(x+h, y+\eta) = \left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x, y) + \left(\overset{1,0}{\Delta}_h \varphi\right)(x, y) + \left(\overset{0,1}{\Delta}_\eta \varphi\right)(x, y) + \varphi(x, y). \qquad (2.1)$$

Evrewhere in the sequel by C, C_1, C_2 etc we denote positive constants which may different values in different occurrences, and even in the same line.

Let $\rho(x, y)$ be a non-negative function on Q (we will only deal with degenerate weights $\rho(x, y) = \rho(x)\rho(y)$).

Below we follow some technical estimations suggested in [1] for the case of one-dimensional Riemann - Liouville fractional integrals. We denote

$$B(x, y; t, \tau) = \frac{\rho(x, y) - \rho(t, \tau)}{\rho(t, \tau)(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}, \qquad (2.2)$$

where $0 < \alpha, \beta < 1$; $0 < t < x < b$, $0 < \tau < y < d$. In the case $\rho(x, y) = \rho(x)\rho(y)$ we have

$$B(x, y; t, \tau) = B_1(x,t)B_2(y,\tau) + \frac{B_1(x,t)}{(y-\tau)^{1-\beta}} + \frac{B_2(y,\tau)}{(x-t)^{1-\alpha}}, \qquad (2.3)$$

where

$$B_1(x,t) = \frac{\rho_1(x) - \rho_1(t)}{\rho_1(t)(x-t)^{1-\alpha}}, \quad B_2(y,\tau) = \frac{\rho_2(y) - \rho_2(\tau)}{\rho_2(\tau)(y-\tau)^{1-\beta}}.$$

Let also

$$D_1(x, h, t) = B_1(x+h, t) - B_1(x, t), \quad t, x, x+h \in [0, b],\ h > 0;$$

$$D_2(y, \eta, t) = B_2(y+\eta, \tau) - B_2(y, \tau), \quad \tau, y, y+\eta \in [0, d],\ \eta > 0.$$

Lemma 2.1. ([1]) Let $\rho_1(x) = x^\mu$, $\mu \in \mathbf{R}^1$, $0 < \alpha < 1$. Then

$$|B_1(x, t)| \le C\left(\frac{x}{t}\right)^{\max(\mu-1,0)} \frac{(x-t)^\alpha}{t}, \qquad (2.4)$$

$$|D_1(x, h, t)| \le C\left(\frac{x+h}{t}\right)^{\max(\mu-1,0)} \frac{h}{t(x+h-t)^{1-\alpha}}. \qquad (2.5)$$

Similar estimates hold for $B_2(y, \tau)$ and $D_2(y, \eta, \tau)$ with $\rho_2(y) = y^\nu$.

Remark 2.1. All the weighted estimations of fractional integrals in the sequel are based on inequalities (2.4)-(2.5). Note that the right - hand sides of these inequalities have the exponent $\max(\mu-1, 0)$, which means that in the proof it suffices to consider only the case $\mu \ge 1$, evaluations of $\mu < 1$ being the same as for $\mu = 1$.

Now we introduce the characteristics

$$\overset{0,1}{\omega}(\varphi; 0, \sigma) = \sup_x \sup_{\eta \in (0,\sigma]} \left|\left(\overset{0,1}{\Delta}_\eta \varphi\right)(x, y)\right|$$ - are partial modulus of continuity of the first order, and a

$\overset{1,1}{\omega}(\varphi;\delta,\sigma)=\sup\limits_{x,y}\sup\limits_{\substack{0<h\le\delta\\0<\eta\le\sigma}}\left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|$ is a mixed modulus of continuity of order (1,1).

Definition 2.1. Let function $\varphi(x)$ is a bounded on $[a,b]$. The modulus of continuity of $\varphi(x)$ is the expression

$$\omega(\varphi;\delta)=\sup_{x_1,x_2\in[a,b]\ |x_1-x_2|\le\delta}|\varphi(x_1)-\varphi(x_2)|,$$

is defined for all δ that satisfy the condition $0<\delta\le b-a$.

Definition 2.2. A function $\omega(\delta)$ $(0<\delta\le b-a)$ is called a modulus of continuity if it satisfies conditions

1) $\lim\limits_{\delta\to0}\omega(\delta)=0$;
2) $\omega(\delta)$ is almost increasing on $(0,b-a]$;
3) $\omega(\delta_1+\delta_2)\le\omega(\delta_1)+\omega(\delta_2)$;
4) $\omega(\delta)$ is function continuous in δ on $(0,b-a]$

Definition 2.3. We denote by Φ^1 the class of functions $\omega(\delta)$ defined on $(0,b-a]$, and satisfying conditions

a) $\omega(\delta)$ is a modulus of continuity
b) $\int\limits_0^{\delta}\frac{\omega(t)}{t}dt\le C\omega(\delta)$;
c) $\delta\int\limits_{\delta}^{b-a}\frac{\omega(t)}{t^2}dt\le C\omega(\delta)$;
d) $\omega'(\delta)\sim\frac{\omega(\delta)}{\delta}$.

It follows from the definition $\overset{1,1}{\omega}(\varphi;\delta,\sigma)$ that this function belongs to Φ^1 each of the variables. In addition, we note the inequality

$$\overset{1,1}{\omega}(\varphi;\delta,\sigma)\le 2\min\left\{\overset{1,0}{\omega}(\varphi;\delta,0),\ \overset{0,1}{\omega}(\varphi;0,\sigma)\right\} \qquad (2.6)$$

Definition 2.4. We denote by $\Phi^{1,1}(Q)$ the class of functions of two variables $\omega(\delta,\sigma)$ satisfying conditions:

1) $\omega(\delta,\sigma)$ in δ for any fixed σ;
2) $\omega(\delta,\sigma)$ in σ for any fixed δ.

We call this class the class of mixed modulus of continuity of the first order of continuous functions of two variables.

The following statements is known, begin first proved in [2]-[4] (see also [6], p. 197). However, here we give a sketch of the proof of this lemma, in order to compose the representation of lightness for the two-dimensional case. Consider the one-dimensional fractional Riemann-Liouville integral

$$\left(I_{0+}^{\alpha}\varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int\limits_0^x\frac{\varphi(t)}{(x-t)^{1-\alpha}}dt,\ x>0, 0<\alpha<1. \qquad (2.7)$$

Theorem 2.1. Let $\varphi(x)$ be continuous on $[0,b]$ and let $\varphi(0)=0$. For the fractional integral (2.7), the estimate

$$\omega\left(I_{0+}^{\alpha}\varphi,h\right)\le Ch\int\limits_h^b\frac{\omega(\varphi,t)}{t^{2-\alpha}}dt \qquad (2.8)$$

is valid.

Proof. Representing (2.7) as

$$\left(I_{0+}^{\alpha}\varphi\right)(x)=\frac{\varphi(0)}{\Gamma(\alpha)}\int\limits_0^x\frac{dt}{(x-t)^{1-\alpha}}+\frac{1}{\Gamma(\alpha)}\int\limits_0^x\frac{\varphi(t)-\varphi(0)}{(x-t)^{1-\alpha}}dt= = A_1(x)+A_2(x)$$

Let $h>0;\ x,x+h\in[0,b]$. We have

$$A_2(x+h)-A_2(x)=\frac{\varphi(x)-\varphi(0)}{\Gamma(1+\alpha)}\left[(x+h)^{\alpha}-x^{\alpha}\right]+ +\frac{1}{\Gamma(\alpha)}\int\limits_0^h\frac{\varphi(x+t)-\varphi(t)}{(h-t)^{1-\alpha}}dt+ +\frac{1}{\Gamma(\alpha)}\int\limits_0^x[\varphi(x-t)-\varphi(t)]\left[(h+t)^{\alpha-1}-t^{\alpha-1}\right]dt=\Delta_1+\Delta_2+\Delta_3.$$

We have: $|\Delta_1|\le C\omega(\varphi;x)\left|(x+h)^{\alpha}-x^{\alpha}\right|$. In the case $x\le h$ we have $|\Delta_1|\le Ch^{\alpha}\omega(\varphi;h)$. Let $x\ge h$. Then

$$|\Delta_1|\le C\omega(\varphi;x)x^{\alpha}\left[\left(1+\frac{h}{x}\right)^{\alpha}-1\right]\le C\frac{\omega(\varphi;x)}{x^{1-\alpha}}h. \qquad (2.9)$$

Since

$$Cx^{\alpha-1}\omega(\varphi;x)\le\omega(\varphi;x)\int\limits_x^b t^{\alpha-2}dx\le\int\limits_x^b\frac{\omega(\varphi;t)}{t^{2-\alpha}}dt\le\int\limits_h^b\frac{\omega(\varphi;t)}{t^{2-\alpha}}dt.$$

It follows from (2.9) that

$$|\Delta_1|\le Ch\int\limits_h^b\frac{\omega(\varphi;t)}{t^{2-\alpha}}dt.$$

Further,

$$|\Delta_2|\le\int\limits_0^h\frac{\omega(\varphi;t)}{(h-t)^{1-\alpha}}dt=h^{\alpha}\int\limits_0^1\frac{\omega(\varphi;h\xi)}{(1-\xi)^{1-\alpha}}d\xi\le Ch^{\alpha}\omega(\varphi;h),$$

with $C=\int\limits_0^1(1-\xi)^{\alpha-1}d\xi$. To estimate Δ_3 we distinguish the case 1) $x\ge h$ and 2) $x\le h$. In the first case

$$|\Delta_3|\le C\left[\int\limits_0^h\omega(f,t)\left[t^{\alpha-1}-(h+t)^{\alpha-1}\right]dt+ +\int\limits_h^x\omega(f,t)\left[t^{\alpha-1}-(h+t)^{\alpha-1}\right]dt\right]\le \le C_2\left[h^{\alpha}\omega(f,h)+h\int\limits_h^b\frac{\omega(f,t)}{t^{2-\alpha}}dt\right].$$

Obviously in the second case $|\Delta_3|\le C_1h^{\alpha}\omega(f;h)$.

Estimates for $\Delta_1,\Delta_2,\Delta_3$ lead to (2.8) if we take into account the fact that $h^{\alpha}\omega(\varphi;h)$ is dominated by the right-hand side of (2.8). The latter is easily obtained in view of the monotonicity of the function $\omega(\varphi;t)$.

To obtain estimates of the Zygmund type in the weighted case, we use the notation and the proof scheme from [1].

Theorem 2.2. Let $\rho(x)=x^{\mu}, 0\le\mu<2-\alpha$. If the function $f(x),\ x\in[0,b]$ satisfies the condition:

1) $\rho(x)f(x)\in C_{[0,b]}$ and $\rho(x)f(x)|_{x=0}=0$;
2) the integral $\int\limits_0^b\frac{\omega(\rho f,t)}{t^{\gamma}}dt$ converges for $\gamma=\max(1,\mu)$.

Then estimates of the Zygmund type

$$\omega(\rho I_{0+}^{\alpha}f,h)\le C\left(h^{\alpha+\gamma-1}\int\limits_0^h\frac{\omega(\rho f,t)}{t^{\gamma}}dt+h\int\limits_h^b\frac{\omega(\rho f,t)}{t^{2-\alpha}}dt\right). \qquad (2.10)$$

Proof. We denote this $g(x)=\rho(x)f(x)$. We have

$$(\rho I_{0+}^{\alpha}f)(x)=\left(I_{0+}^{\alpha}g\right)(x)+\left(J_{0+}^{\alpha}g\right)(x),\ \left(J_{0+}^{\alpha}g\right)(x)=\frac{1}{\Gamma(\alpha)}\int\limits_0^xB(x,t)g(t)dt.$$

Here the estimates for $\left(I_{0+}^{\alpha}g\right)(x)$ are solved in Theorem 2.1.

Now consider the difference

$$\left(J_{0+}^{\alpha}g\right)(x+h)-\left(J_{0+}^{\alpha}g\right)(x)=F_1(x,h)+F_2(x,h),$$

where

$$F_1(x,h)=\int_x^{x+h}B(x+h,t)g(t)dt,\quad F_2(x,h)=\int_a^x D(x,h,t)g(t)dt.$$

Taking into account Remark 1.1, we consider only the case $1\le\mu<2-\alpha$. From (2.4) we have

$$|F_1|\le C\int_x^{x+h}\left(\frac{x+h}{t}\right)^{\mu-1}\frac{(x+h-t)^{\alpha}}{t}\omega(g,t)dt.$$

If $x\le h$, then

$$|F_1|\le Ch^{\mu+\alpha-1}\int_x^{x+h}\frac{\omega(g;t)}{t^{\mu}}dt.$$

Using the property of almost decreasing $\frac{\omega(g;t)}{t}$, we obtain

$$|F_1|\le Ch^{\mu+\alpha-1}\int_x^{x+h}\frac{\omega(g;t-x)}{(t-x)^{\mu}}dt=Ch^{\mu+\alpha-1}\int_0^h\frac{\omega(g;t)}{t^{\mu}}dt.$$

If $x>h$, then

$$|F_1|\le Ch^{\alpha}(x+h)^{\mu-1}\int_x^{x+h}\frac{\omega(g;t)}{t^{\mu}}dt=C\frac{h^{\alpha}}{(x+h)^{\mu-1}}\int_0^h\frac{\omega(g;x+t)}{(x+t)^{\mu}}dt\le$$

$$\le Ch^{\alpha}x^{\mu-1}\int_0^h\frac{\omega(g;x+t)}{(x+t)^{\mu-1}}\frac{dt}{x+t}\le Ch^{\alpha}\int_0^h\frac{\omega(g;x+t)}{x+t}dt.$$

Further, it is clear that

$$|F_1|\le Ch^{\alpha}\int_0^h\frac{\omega(g;t)}{t}dt.$$

Collecting the estimates F_1, we obtain the inequality for $0\le\mu<2-\alpha$

$$|F_1|\le Ch^{\alpha+\gamma-1}\int_0^h\frac{\omega(g;t)}{t^{\gamma}}dt,\quad \gamma=\max(1,\mu).$$

We pass to the estimate F_2. Using the estimate (2.5), we obtain

$$|F_2|\le Ch\int_0^x\left(\frac{x+h}{t}\right)^{\mu-1}\frac{\omega(g;t)}{(x+h-t)^{1-\alpha}t}dt. \qquad (2.11)$$

When $h\ge x$,

$$|F_2|\le Ch^{\alpha+\mu-1}\int_0^x\frac{\omega(g;t)}{t^{\mu}}dt\le Ch^{\alpha+\mu-1}\int_0^h\frac{\omega(g;t)}{t^{\mu}}dt.$$

If $h<x$, then, we represent the right-hand side of (2.11) as a sum of three terms:

$$F_2'=Ch\int_0^h\left(\frac{x+h}{t}\right)^{\mu-1}\frac{\omega(g;t)}{(x+h-t)^{1-\alpha}}\frac{dt}{t},$$

$$F_2''=Ch\int_h^{\frac{1}{2}(x+h)}\left(\frac{x+h}{t}\right)^{\mu-1}\frac{\omega(g;t)}{(x+h-t)^{1-\alpha}}\frac{dt}{t},$$

$$F_2'''=Ch\int_{\frac{1}{2}(x+h)}^{x}\left(\frac{x+h}{t}\right)^{\mu-1}\frac{\omega(g;t)}{(x+h-t)^{1-\alpha}}\frac{dt}{t}.$$

Then $|F_2|\le F_2'+F_2''+F_2'''$.

For the term F_2' the relations are valid $x+h\le 2(x+h-t)$, therefore

$$F_2'\le Ch\int_0^h\frac{\omega(g;t)dt}{t^{\mu}(x+h-t)^{2-\alpha-\mu}}\le Ch^{\alpha+\mu-1}\int_0^h\frac{\omega(g;t)}{t^{\mu}}dt.$$

For the summand F_2'' we have $2t\le x+h$, so $1\le\mu<2-\alpha$ we obtain the estimate

$$F_2''\le Ch\int_h^{\frac{1}{2}(x+h)}\frac{\omega(g;t)}{t^{\mu}\left(\frac{x+h}{2}\right)^{2-\mu-\alpha}}dt\le Ch\int_h^b\frac{\omega(g;t)}{t^{2-\alpha}}dt.$$

We estimate the term F_2'''. Here $t\ge x+h-t$, therefore $\frac{\omega(g;t)}{t}\le C\frac{\omega(g;x+h-t)}{x+h-t}$, it follows that

$$F_2'''\le Ch\int_{\frac{1}{2}(x+h)}^{x}\frac{\omega(g;x+h-t)}{(x+h-t)^{2-\alpha}}dt.$$

Because, $x+h\le 2t$. Having made the change $\xi=x+h-t$ and going back to the variable t, we get

$$F_2'''\le Ch\int_h^b\frac{\omega(g;t)}{t^{2-\alpha}}dt.$$

From the estimates F_2', F_2'', F_2''' follows when $h<x$

$$|F_2|\le C\left(h^{\mu+\alpha-1}\int_0^h\frac{\omega(g;t)}{t^{\mu}}dt+h\int_h^b\frac{\omega(g;t)}{t^{2-\alpha}}dt\right).$$

Thus, when $0\le\mu<2-\alpha$

$$|F_2|\le C\left(h^{\gamma+\alpha-1}\int_0^h\frac{\omega(g;t)}{t^{\gamma}}dt+h\int_h^b\frac{\omega(g;t)}{t^{2-\alpha}}dt\right),\quad \gamma=\max(1,\mu),$$

which completes the proof.

3. Zygmund type estimates

Theorem 3.1. Let $\varphi\in\mathbf{C}(Q)$ and $\varphi(x,y)=|_{x=0,\,y=0}=0$. Then for (1.1) we have estimates of the Zygmund type

$$\overset{1,1}{\omega}(f;h,\eta)\le C_1h\eta\int_h^b\int_{\eta}^d\frac{\overset{1,1}{\omega}(\varphi;t,\tau)}{t^{2-\alpha}\tau^{2-\beta}}dtd\tau. \qquad (3.1)$$

$$\overset{1,0}{\omega}(f;h,0)\le C_2h\int_h^b\frac{\overset{1,1}{\omega}(\varphi;t,d)}{t^{2-\alpha}}dt,$$

$$\overset{0,1}{\omega}(f;0,\eta)\le C_3\eta\int_{\eta}^d\frac{\overset{1,1}{\omega}(\varphi;b,\tau)}{\tau^{2-\beta}}d\tau. \qquad (3.2)$$

Proof. Using the identity (2.1), we represent the integral (1.1) in the form

$$\left(I_{0+,0+}^{\alpha,\beta}\varphi\right)(x,y)=\frac{\varphi(0,0)x^{\alpha}y^{\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)}+\frac{x^{\alpha}\psi_2(y)}{\Gamma(1+\alpha)}+\frac{y^{\beta}\psi_1(x)}{\Gamma(1+\beta)}+\psi(x,y),$$

where

$$\psi_1(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{\varphi(t,0)-\varphi(0,0)}{(x-t)^{1-\alpha}}dt,\quad \psi_2(y)=\frac{1}{\Gamma(\beta)}\int_0^y\frac{\varphi(0,\tau)-\varphi(0,0)}{(y-\tau)^{1-\beta}}d\tau,$$

$$\psi(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y\frac{\left(\overset{1,1}{\Delta}_{t,\tau}\varphi\right)(0,0)}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}dtd\tau.$$

Let $h>0,\ x, x+h\in[0,b]$. Consider the difference

$$\left(\overset{1,0}{\Delta}_h f\right)(x,y)=\frac{(x+h)^{\alpha}-x^{\alpha}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\frac{g(x,y-\tau)}{\tau^{1-\beta}}d\tau+$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{-h}^0\int_0^y\frac{g(x-t,y-\tau)-g(x,y-\tau)}{\tau^{1-\beta}(t+h)^{1-\alpha}}dtd\tau+$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y \frac{g(x-t,y-\tau)-g(x,y-\tau)}{\tau^{1-\beta}}\left|(t+h)^{\alpha-1}-t^{\alpha-1}\right|dtd\tau .$$

The following inequality is valid

$$\left|\left(\overset{1,0}{\Delta}_h f\right)(x,y)\right|\le C\left(|(x+h)^\alpha-x^\alpha|\int_0^y\frac{\overset{1,1}{\omega}(\varphi;x,y-\tau)}{\tau^{1-\beta}}d\tau+\right.$$
$$+\int_{-h}^0\int_0^y \frac{\overset{1,1}{\omega}(\varphi;t,y-\tau)}{(h+t)^{1-\alpha}\tau^{1-\beta}}dtd\tau+$$
$$\left.+\int_0^x\int_0^y\overset{1,1}{\omega}(\varphi;t,y-\tau)|(h+t)^{\alpha-1}-t^{\alpha-1}|\tau^{\beta-1}dtd\tau\right).$$

We make use of (2.1) and obtain

$$\left|\left(\overset{1,0}{\Delta}_h f\right)(x,y)\right|\le C_1\left(|(x+h)^\alpha-x^\alpha|\overset{1,1}{\omega}(\varphi;x,d)+\int_{-h}^0\frac{\overset{1,1}{\omega}(\varphi;t,d)}{(h+t)^{1-\alpha}}dt+\right.$$
$$\left.+\int_0^x\overset{1,1}{\omega}(\varphi;t,d)|(h+t)^{\alpha-1}-t^{\alpha-1}|dt\right).$$

Using the estimates $\Delta_1, \Delta_2, \Delta_3$ in the proof of Theorem 2.1, it is easy to obtain

$$|f(x,y+\eta)-f(x,y)|\le C_2\eta\int_\eta^{d-c}\frac{\overset{1,1}{\omega}(\varphi;b-a,\tau)}{\tau^{2-\beta}}d\tau. \qquad (3.3)$$

Similarly, we can obtain the estimate

$$|f(x,y+\eta)-f(x,y)|\le C_2\eta\int_\eta^{d}\frac{\overset{1,1}{\omega}(\varphi;b,\tau)}{\tau^{2-\beta}}d\tau. \qquad (3.4)$$

From (3.3) and (3.4) follows the inequalities (3.2).

Let $h,\eta>0$ and $x,x+h\in[a,b]$, $y,y+\eta\in[c,d]$. Consider the difference

$$\left(\overset{1,1}{\Delta}_{h,\eta} f\right)(x,y)=\left(\overset{1,1}{\Delta}_{h,\eta}\psi\right)(x,y)=\sum_{k=1}^{9} T_k:=$$
$$:=\frac{g(x,y)}{\Gamma(1+\alpha)\Gamma(1+\beta)}\left[(x+h)^\alpha-x^\alpha\right][(y+\eta)^\beta-y^\beta]+$$
$$+\frac{(y+\eta)^\beta-y^\beta}{\Gamma(\alpha)\Gamma(1+\beta)}\int_{-h}^0\frac{g(x-t,y)-g(x,y)}{(t+h)^{1-\alpha}}dt+$$
$$+\frac{(x+h)^\alpha-x^\alpha}{\Gamma(1+\alpha)\Gamma(\beta)}\int_{-\eta}^0\frac{g(x,y-\tau)-g(x,y)}{(\tau+\eta)^{1-\beta}}d\tau+$$
$$+\frac{(y+\eta)^\beta-y^\beta}{\Gamma(\alpha)\Gamma(1+\beta)}\int_0^x[g(x-t,y)-g(x,y)]\left[(t+h)^{\alpha-1}-t^{\alpha-1}\right]dt+$$
$$+\frac{(x+h)^\alpha-x^\alpha}{\Gamma(1+\alpha)\Gamma(\beta)}\int_0^y[g(x,y-\tau)-g(x,y)]\left[(\tau+\eta)^{\beta-1}-\tau^{\beta-1}\right]d\tau+$$
$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{-h}^0\int_{-\eta}^0\frac{\left(\overset{1,1}{\Delta}_{-t,-\tau}g\right)(x,y)}{(h+t)^{1-\alpha}(\eta+\tau)^{1-\beta}}dtd\tau+$$
$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{-h}^0\int_0^y\frac{\left(\overset{1,1}{\Delta}_{-t,-\tau}g\right)(x,y)}{(h+t)^{1-\alpha}}\left[(\tau+\eta)^{\beta-1}-\tau^{\beta-1}\right]dtd\tau+$$
$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_{-\eta}^0\frac{\left(\overset{1,1}{\Delta}_{-t,-\tau}g\right)(x,y)}{(\eta+\tau)^{1-\beta}}\left[(t+h)^{\alpha-1}-t^{\alpha-1}\right]dtd\tau+$$
$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y\left(\overset{1,1}{\Delta}_{-t,-\tau}g\right)(x,y)\left[(t+h)^{\alpha-1}-t^{\alpha-1}\right]\times$$
$$\left[(\tau+\eta)^{\beta-1}-\tau^{\beta-1}\right]dtd\tau .$$

The inequality is valid

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta} f\right)(x,y)\right|\le C\left(\overset{1,1}{\omega}(\varphi;x,y)\,|(x+h)^\alpha-x^\alpha|\,|(y+\eta)^\beta-y^\beta|+\right.$$
$$+|(y+\eta)^\beta-y^\beta|\int_0^h\frac{\overset{1,1}{\omega}(\varphi;t,y)}{(h-t)^{1-\alpha}}dt+|(x+h)^\alpha-x^\alpha|\int_0^\eta\frac{\overset{1,1}{\omega}(\varphi;x,\tau)}{(\eta-\tau)^{1-\beta}}d\tau+$$
$$+|(y+\eta)^\beta-y^\beta|\int_0^x\overset{1,1}{\omega}(\varphi;t,y)|(h+t)^{\alpha-1}-t^{\alpha-1}|dt+$$
$$+|(x+h-a)^\alpha-(x-a)^\alpha|\int_0^{y-c}\overset{1,1}{\omega}(\varphi;x-a,\tau)|(\eta+\tau)^{\beta-1}-\tau^{\beta-1}|d\tau+$$
$$+\int_0^h\int_0^\eta\frac{\overset{1,1}{\omega}(\varphi;t,\tau)}{(h-t)^{1-\alpha}(\eta-\tau)^{1-\beta}}dtd\tau+$$
$$+\int_0^h\int_0^y\overset{1,1}{\omega}(\varphi;t,\tau)(h-t)^{\alpha-1}|(\eta+\tau)^{\beta-1}-\tau^{\beta-1}|dtd\tau+$$
$$+\int_0^x\int_0^\eta\overset{1,1}{\omega}(\varphi;t,\tau)|(h+t)^{\alpha-1}-t^{\alpha-1}|(\eta-\tau)^{\beta-1}dtd\tau+$$
$$\left.+\int_0^x\int_0^y\overset{1,1}{\omega}(\varphi;t,\tau)|(h+t)^{\alpha-1}-t^{\alpha-1}|\,|(\eta+\tau)^{\beta-1}-\tau^{\beta-1}|dtd\tau\right).$$

Each term of this inequality is estimated in the standard way and one can obtain

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta} f\right)(x,y)\right|\le C_3h\eta\int_h^b\int_\eta^d\frac{\overset{1,1}{\omega}(\varphi;t,\tau)}{t^{2-\alpha}\tau^{2-\beta}}dtd\tau,$$

from which inequality (3.1) follows.

Theorem 3.2. Let $\rho(x,y)=\rho(x)\rho(y)=x^\mu y^\nu$, $0\le\mu<2-\alpha$, $0\le\nu<2-\beta$. If the function $\varphi(x,y)\in Q$ satisfies the following conditions:

1) $\varphi_0(x,y)=\rho(x,y)\varphi(x,y)\in \mathbf{C}(Q)$ and $\varphi_0(x,y)|_{x=0,\,y=0}=0$;

2) $\int_0^{b-a}\int_0^{d-c}\frac{\overset{1,1}{\omega}(\varphi_0;t,\tau)}{t^\gamma\tau^\lambda}dtd\tau$ the integral converges for $\gamma=\max\{1,\mu\}$, $\lambda=\max\{1,\nu\}$. Then the following estimates of Zygmund type are valid

$$\overset{1,0}{\omega}(\rho f;h,0)\le C_1\left[h^{\alpha+\gamma-1}\int_0^h\frac{\overset{1,1}{\omega}(\rho\varphi,t,d)}{t^\gamma}dt+\right.$$
$$\left.+h\int_h^b\frac{\overset{1,1}{\omega}(\rho\varphi,t,d)}{t^{2-\alpha}}dt\right], \qquad (3.5)$$

$$\overset{0,1}{\omega}(\rho f;0,\eta)\le C_2\left[\eta^{\beta+\lambda-1}\int_0^\eta\frac{\overset{1,1}{\omega}(\rho\varphi,b,\tau)}{\tau^\lambda}d\tau+\right.$$
$$\left.+\eta\int_\eta^d\frac{\overset{1,1}{\omega}(\rho\varphi,b,\tau)}{\tau^{2-\beta}}d\tau\right], \qquad (3.4)$$

$$\overset{1,1}{\omega}(\rho f;h,\eta)\le C_3\Bigg[h^{\alpha+\gamma-1}\eta^{\beta+\lambda-1}\int_0^h\int_0^\eta\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^\gamma\tau^\lambda}dtd\tau+$$

$$+h\eta^{\beta+\lambda-1}\int_h^b\int_0^\eta\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^{2-\alpha}\tau^\lambda}dtd\tau+h^{\alpha+\gamma-1}\eta\int_0^h\int_\eta^d\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^\gamma\tau^{2-\beta}}dtd\tau+$$

$$+h\eta\int_h^b\int_\eta^d\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^{2-\alpha}\tau^{2-\beta}}dtd\tau\Bigg]. \qquad (3.5)$$

Proof. By Remark 2.1, it suffices to deal with the case $\mu,\nu\ge 1$. Let $\varphi\in\tilde H_0^\omega(\rho)$, so that $\varphi_0(x,y)=\varphi(x,y)\rho(x,y)$, where $\varphi_0(x,y)\in\tilde H_0^\omega(\rho)$ and $\varphi_0(x,y)\big|_{x=0,\ y=0}=0$. For

$$G(x,y):=\int_0^x\int_0^y\frac{\rho(x,y)\varphi_0(t,\tau)dtd\tau}{\rho(t,\tau)(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}.$$

We represent $G(x,y)$ in the form

$$G(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left(\int_0^x\int_0^y\frac{\varphi_0(t,\tau)dtd\tau}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}+\right.$$

$$\left.+\int_0^x\int_0^y B(x,y;t,\tau)\varphi_0(t,\tau)dtd\tau\right)=G_1(x,y)+G_2(x,y).$$

Here the question of the estimation of the modulus of continuity for the first term is solved by us in Theorem 3.1. Therefore, inequalities

$$\overset{1,0}{\omega}(G_1;h,0)\le C_1\left(h^\alpha\,\overset{1,1}{\omega}(\varphi_0;h,d)+h\int_h^b\frac{\overset{1,1}{\omega}(\varphi_0;t,d)}{t^{2-\alpha}}dt\right), \qquad (3.6)$$

$$\overset{0,1}{\omega}(G_1;0,\eta)\le C_2\left(\eta^\beta\,\overset{1,1}{\omega}(\varphi_0;b,\eta)+\eta\int_\eta^d\frac{\overset{1,1}{\omega}(\varphi_0;b,\tau)}{\tau^{2-\beta}}d\tau\right), \qquad (3.7)$$

$$\overset{1,1}{\omega}(G_1;h,\eta)\le C_3\left(h^\alpha\eta^\beta\,\overset{1,1}{\omega}(\varphi_0;h,\eta)+h\eta^\beta\int_h^b\frac{\overset{1,1}{\omega}(\varphi_0;t,\eta)}{t^{2-\alpha}}dt+\right.$$

$$\left.+h^\alpha\eta\int_\eta^d\frac{\overset{1,1}{\omega}(\varphi_0;h,\tau)}{\tau^{2-\beta}}d\tau+h\eta\int_h^b\int_\eta^d\frac{\overset{1,1}{\omega}(\varphi_0;t,\tau)}{t^{2-\alpha}\tau^{2-\beta}}dtd\tau\right). \qquad (3.8)$$

To estimate the term $G_2(x,y)$, we note that the weight being degenerate, we have

$$\rho(x,y)-\rho(t,\tau)=[\rho(x)-\rho(t)][\rho(y)-\rho(\tau)]+\rho(\tau)[\rho(x)-\rho(t)]+$$
$$+\rho(t)[\rho(y)-\rho(\tau)],$$

which leads to the following representation

$$G_2(x,y)=\int_0^x\int_0^y B_1(x,t)B_2(y,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_0^x\int_0^y B_1(x,t)\varphi_0(t,\tau)(y-\tau)^{\beta-1}dtd\tau+$$

$$+\int_0^x\int_0^y B_2(y,\tau)\varphi_0(t,\tau)(x-t)^{\alpha-1}dtd\tau.$$

where the notation (2.3) has been used. For the difference $\left(\overset{1,0}{\Delta}_h G_2\right)(x,y)$ with $h>0$ and $x,x+h\in(0,b)$, we have

$$\left(\overset{1,0}{\Delta}_h G_2\right)(x,y)=\int_x^{x+h}\int_0^y B_1(x+h,t)B_2(y,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_0^x\int_0^y D_1(x,h,t)B_2(y,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_0^y B_1(x+h,t)\frac{\varphi_0(t,\tau)}{(y-\tau)^{1-\beta}}dtd\tau+$$

$$+\int_0^x\int_0^y D_1(x,h,t)\frac{\varphi_0(t,\tau)}{(y-\tau)^{1-\beta}}dtd\tau+\int_x^{x+h}\int_0^y\frac{\varphi_0(t,\tau)B_2(y,\tau)}{(x+h-t)^{1-\alpha}}dtd\tau+$$

$$+\int_0^x\int_0^y\varphi_0(t,\tau)\left[(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}\right]B_2(y,\tau)dtd\tau.$$

Since $\varphi_0(x,0)=0$, then the inequality

$$\left|\left(\overset{1,0}{\Delta}_h G_2\right)(x,y)\right|\le\int_x^{x+h}\int_0^y|B_1(x+h,t)||B_2(y,\tau)|\,\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_0^x\int_0^y|D_1(x,h,t)||B_2(y,\tau)|\,\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$\int_x^{x+h}\int_0^y|B_1(x+h,t)|\frac{\overset{1,1}{\omega}(\varphi_0;t,\tau)}{(y-\tau)^{1-\beta}}dtd\tau+$$

$$+\int_0^x\int_0^y|D_1(x,h,t)|\frac{\overset{1,1}{\omega}(\varphi_0;t,\tau)}{(y-\tau)^{1-\beta}}dtd\tau+$$

$$\int_x^{x+h}\int_0^y\frac{\overset{1,1}{\omega}(\varphi_0;t,\tau)}{(x+h-t)^{1-\alpha}}|B_2(y,\tau)|dtd\tau+$$

$$+\int_0^x\int_0^y\overset{1,1}{\omega}(\varphi_0;t,\tau)\,|(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}|\,|B_2(y,\tau)|dtd\tau.$$

We make use of (2.1) and obtain

$$|G_2(x+h,y)-G_2(x,y)|\le\int_x^{x+h}|B_1(x+h,t)|\,\overset{1,1}{\omega}(\varphi_0;t,d)dt+$$

$$+\int_0^x|D_1(x,h,t)|\,\overset{1,1}{\omega}(\varphi_0;t,d)dt+\int_x^{x+h}|B_1(x+h,t)|\,\overset{1,1}{\omega}(\varphi_0;t,d)dt+$$

$$+\int_0^x|D_1(x,h,t)|\,\overset{1,1}{\omega}(\varphi_0;t,d)dt+\int_x^{x+h}\frac{\overset{1,1}{\omega}(\varphi_0;t,d)}{(x+h-t)^{1-\alpha}}dt+$$

$$+\int_0^x\overset{1,1}{\omega}(\varphi_0;t,d)\,|(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}|\,dt.$$

From the estimates $\Delta_1,\Delta_2,\Delta_3$ of Theorem 2.1 and from the estimates F_1,F_2 in Theorem 2.2, one can easily verify the validity of inequality

$$\left|\left(\overset{1,0}{\Delta}_h G_2\right)(x,y)\right|\le C_1\Bigg[h^{\alpha+\gamma-1}\int_0^h\frac{\overset{1,1}{\omega}(\rho\varphi;t,d)}{t^\gamma}dt$$

$$+h\int_h^{b-a}\frac{\overset{1,1}{\omega}(\rho\varphi;t,d-c)}{t^{2-\alpha}}dt\Bigg], \qquad (3.9)$$

где $\gamma=\max(1,\mu)$.

The estimate

$$\left|\left(\overset{0,1}{\Delta}_{\eta} G_2\right)(x,y)\right| \le C_2\left[\eta^{\beta+\lambda-1}\int_0^{\eta}\frac{\overset{1,1}{\omega}(\rho\varphi;b,\tau)}{\tau^{\lambda}}d\tau\right.$$

$$\left.+\eta\int_{\eta}^{d}\frac{\overset{1,1}{\omega}(\rho\varphi;b,\tau)}{\tau^{2-\beta}}d\tau\right], \qquad (3.10)$$

is symmetrically obtained, where $\lambda=\max(1,\nu)$.

For the mixed difference $\left(\overset{1,1}{\Delta}_{h,\eta} G_2\right)(x,y)$ with $h,\eta>0$ and $x,x+h\in[0,b],\ y,y+\eta\in[0,d]$ the appropriate representation leading to the separate evaluation in each variable without losses in another variable is as follows:

$$\left(\overset{1,1}{\Delta}_{h,\eta} G_2\right)(x,y)=\int_x^{x+h}\int_y^{y+\eta}B_1(x+h,t)B_2(y+\eta,\tau)\varphi_0(t,\tau)d\tau dt+$$

$$+\int_0^x\int_0^y D_1(x,h,t)D_2(y,\eta,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_0^y B_1(x+h,t)D_2(y,\eta,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_0^x\int_y^{y+\eta} D_1(x,h,t)B_2(y+\eta,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_y^{y+\eta}\frac{B_1(x+h,t)}{(y+\eta-\tau)^{1-\beta}}\varphi_0(t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_0^y B_1(x+h,t)\left[(y+\eta-\tau)^{\beta-1}-(y-\tau)^{\beta-1}\right]\varphi_0(t,\tau)dtd\tau+$$

$$+\int_0^x\int_y^{y+\eta} D_1(x,h,t)(y+\eta-\tau)^{\beta-1}\varphi_0(t,\tau)dtd\tau+$$

$$+\int_0^x\int_0^y D_1(x,h,t)\left[(y+\eta-\tau)^{\beta-1}-(y-\tau)^{\beta-1}\right]\varphi_0(t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_y^{y+\eta}(x+h-t)^{\alpha-1}B_2(y+\eta,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_0^x\int_y^{y+\eta}\left[(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}\right]B_2(y+\eta,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_0^y(x+h-t)^{\alpha-1}D_2(y,\eta,\tau)\varphi_0(t,\tau)dtd\tau+$$

$$+\int_0^x\int_0^y\left[(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}\right]D_2(y,\eta,\tau)\varphi_0(t,\tau)dtd\tau.$$

The inequality is rightly

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta} G_2\right)(x,y)\right|\le C\left|\int_x^{x+h}\int_y^{y+\eta}B_1(x+h,t)B_2(y+\eta,\tau)\times\right.$$

$$\times\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+\int_0^x\int_0^y D_1(x,h,t)D_2(y,\eta,\tau)\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_0^y B_1(x+h,t)D_2(y,\eta,\tau)\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_0^x\int_y^{y+\eta} D_1(x,h,t)B_2(y+\eta,\tau)\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_y^{y+\eta}\frac{B_1(x+h,t)}{(y+\eta-\tau)^{1-\beta}}\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_0^y B_1(x+h,t)\left[(y+\eta-\tau)^{\beta-1}-(y-\tau)^{\beta-1}\right]\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_0^x\int_y^{y+\eta} D_1(x,h,t)(y+\eta-\tau)^{\beta-1}\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_0^x\int_0^y D_1(x,h,t)\left[(y+\eta-\tau)^{\beta-1}-(y-\tau)^{\beta-1}\right]\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_y^{y+\eta}(x+h-t)^{\alpha-1}B_2(y+\eta,\tau)\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_0^x\int_y^{y+\eta}\left[(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}\right]B_2(y+\eta,\tau)\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$+\int_x^{x+h}\int_0^y(x+h-t)^{\alpha-1}D_2(y,\eta,\tau)\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau+$$

$$\left.+\int_0^x\int_0^y\left[(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}\right]D_2(y,\eta,\tau)\overset{1,1}{\omega}(\varphi_0;t,\tau)dtd\tau\right|.$$

We omit the details of evaluation of each term in the above representation, it is standard via Lemma 2.1 and yields

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta} G_2\right)(x,y)\right|\le C_3\left[h^{\alpha+\gamma-1}\eta^{\beta+\lambda-1}\int_0^h\int_0^{\eta}\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^{\gamma}\tau^{\lambda}}dtd\tau+\right.$$

$$+h\eta^{\beta+\lambda-1}\int_h^b\int_0^{\eta}\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^{2-\alpha}\tau^{\lambda}}dtd\tau+h^{\alpha+\gamma-1}\eta\int_0^h\int_{\eta}^d\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^{\gamma}\tau^{2-\beta}}dtd\tau+$$

$$\left.+h\eta\int_h^b\int_{\eta}^d\frac{\overset{1,1}{\omega}(\rho\varphi;t,\tau)}{t^{2-\alpha}\tau^{2-\beta}}dtd\tau\right], \qquad (3.11)$$

where $\gamma=\max(1,\mu)$ и $\lambda=\max(1,\nu)$.

From the inequalities (3.11), (3.10), (3.9) and (3.6), (3.7), (3.8), we obtain the corresponding estimates (3.3), (3.4) and (3.5).

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